A method to tackle first-order ordinary differential equations with Liouvillian functions in the solution

2002 J. Phys. A: Math. Gen. 35 3899

(http://iopscience.iop.org/0305-4470/35/17/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 3899–3910 PII: S0305-4470(02)27514-9

A method to tackle first-order ordinary differential equations with Liouvillian functions in the solution

L G S Duarte[1], **S E S Duarte**[2,3] **and L A C P da Mota**[1]

[1] Depto de Física Teórica, Instituto de Física, Universidade do Estado do Rio de Janeiro, 20559-900 Rio de Janeiro, RJ, Brazil
[2] Departamento de Engenharia Elétrica, Universidade Estácio de Sá, Rio de Janeiro, RJ, Brazil
[3] Centro Brasileiro de, Pesquisas Físicas, Rio de Janeiro, RJ, Brazil

E-mail: lduarte@dft.if.uerj.br, eduarte@cbpf.br and damota@dft.if.uerj.br

Received 2 August 2001, in final form 1 March 2002
Published 19 April 2002
Online at stacks.iop.org/JPhysA/35/3899

Abstract

The usual Prelle–Singer (PS) approach misses many first-order ordinary differential equations presenting Liouvillian functions in the solution (LFOODEs). We point out why and propose a method extending the PS method to solve a class of these previously unsolved LFOODEs. Although our method does not cover all the LFOODEs, it maintains the semi-decision nature of the usual PS method.

PACS numbers: 02.30.Hq, 02.20.−a, 02.60.Gf

1. Introduction

The problem of solving ordinary differential equations (ODEs) has led, over the years, to a wide range of different methods for their solution. Along with many techniques for calculating tricky integrals, these often occupy a large part of the mathematics syllabuses of university courses in applied mathematics round the world.

The overwhelming majority of these methods are based on the classification of the DE into types for which a method of solution is known, which has resulted in a gamut of methods that deal with specific classes of DEs. This scene changed somewhat at the end of the 19th century when Sophus Lie developed a general method to solve (or at least reduce the order of) ordinary differential equations given their symmetry transformations [1–3]. Lie's method is very powerful and highly general, but first requires that we find the symmetries of the differential equation, which may not be easy to do. Search methods have been developed [4, 5] to extract the symmetries of a given ODE, however these methods are heuristic and cannot guarantee that, if symmetries exist, they will be found.

A big step forward in constructing an algorithm for solving first-order ODEs (FOODEs) analytically was taken in a seminal paper by Prelle and Singer (PS) [6] on autonomous systems of ODEs. Prelle and Singer's problem is equivalent to asking when a FOODE of the form $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$, with M and N polynomials in their arguments, has an elementary solution[4] (roughly speaking, a solution which can be written in terms of a combination of polynomials, logarithms, exponentials and radicals). The great advantage of the PS approach is the fact that, up to a certain degree (see section 2), the method ensures that, if it exists, the elementary solution will be found. Prelle and Singer were not exactly able to construct an algorithm for solving their problem, since they were not able to define the degree bound for the polynomials which might enter into the construction of an integrating factor for the FOODE in question. Though this is important from a theoretical point of view, any practical use of the PS method will have a degree bound imposed by the time necessary to perform the actual calculation needed to handle the evermore complex equations. With this in mind it is possible to say that Prelle and Singer's original method is almost an algorithm, awaiting a theoretical degree bound to turn it algorithmic. Due to its semi-decision nature, the PS approach has motivated many extensions [7–10].

The PS approach is valid for solving first-order differential equations with elementary solutions. Despite this fact, as we will show in this paper, the PS approach solves some first-order differential equations whose solutions are not elementary (i.e., cannot be written in terms of elementary functions). These solutions are written in terms of Liouvillian functions[5] (a generalization of the elementary functions). The purpose of this paper is twofold: first, using previous results due to Prelle and Singer [6, 11], we take another step in the knowledge of the integrating factor structure for LFOODEs, thus understanding why some LFOODEs (with non-elementary solutions) can be solved via the PS approach while others cannot be; then, using this knowledge, we construct a method allowing for the solution of a class of those LFOODEs which are missed by the usual PS approach.

The paper is organized as follows: in section 2, we present a short theoretical introduction to the PS approach; in the following section, we show why the Prelle–Singer approach solves some LFOODEs (with non-elementary solutions) and misses others; in section 4, we present a new result concerning the integrating factor structure for LFOODEs; in section 5, we introduce the main ideas of our method with examples of its application; we finally present our conclusions.

2. The Prelle–Singer procedure

Despite its usefulness in solving FOODEs, the Prelle–Singer procedure is not very well known outside mathematical circles, and so we present a brief overview of the main ideas of the procedure.

Consider the class of FOODEs which can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M(x, y)}{N(x, y)} \tag{1}$$

where $M(x, y)$ and $N(x, y)$ are polynomials with coefficients in the complex field C. Rewriting (1) as a Pfaffian equation, one gets

$$M(x, y)\,\mathrm{d}x - N(x, y)\,\mathrm{d}y = 0. \tag{2}$$

Suppose there is a function $R(x, y)$ such that

$$R(M\,\mathrm{d}x - N\mathrm{d}y) = RM\,\mathrm{d}x - RN\,\mathrm{d}y = \mathrm{d}\beta \tag{3}$$

[4] For a formal definition of elementary function, see [13].
[5] Roughly speaking, Liouvillian functions are built up from rational functions using exponentiation, integration and algebraic functions. For a formal definition see [13].

i.e., the 1-form $(RM)\,\mathrm{d}x - (RN)\,\mathrm{d}y$ is exact. The function R is then called an integrating factor[6] for (1).

From (3) we can conclude that $\partial_x(RN) + \partial_y(RM) = 0$, leading to $N\partial_x R + R\partial_y N + M\partial_y R + R\partial_y M = 0$. Thus, we finally obtain

$$\frac{D[R]}{R} = -(\partial_x N + \partial_y M) \tag{4}$$

where $D \equiv N\partial_x + M\partial_y$.

In [6], Prelle and Singer proved that, if the solution of (1) is written in terms of elementary functions, then, for this FOODE, there exists an integrating factor of the form $R = \prod_i f_i^{n_i}$ where f_i are irreducible polynomials and n_i are nonzero rational numbers. Using this result in (4), we have

$$\begin{aligned}\frac{D[R]}{R} &= \frac{D\left[\prod_i f_i^{n_i}\right]}{\prod_i f_k^{n_k}} = \frac{\sum_i f_i^{n_i-1} n_i D[f_i] \prod_{j\neq i} f_j^{n_j}}{\prod_k f_k^{n_k}}\\ &= \sum_i \frac{f_i^{n_i-1} n_i D[f_i]}{f_i^{n_i}} = \sum_i \frac{n_i D[f_i]}{f_i}.\end{aligned} \tag{5}$$

From (4), plus the fact that M and N are polynomials, they concluded that $D[R]/R$ is a polynomial and that $f_i \mid D[f_i]$[7] [6].

We now have a criterion for choosing the possible f_i (build all the possible divisors (up to a certain degree) of $D[f_i]$) and, using (4) and (5), we have

$$\sum_i \frac{n_i D[f_i]}{f_i} = -(\partial_x N + \partial_y M). \tag{6}$$

If we manage to solve (6) and thereby find n_i, we know the integrating factor for the FOODE and the problem is reduced to a quadrature.

3. The PS method and LFOODEs

In this section, we will comment on the effectiveness of the PS approach to LFOODEs. The Prelle–Singer result [6], as already mentioned, assures that if there exists a solution written in terms of elementary functions to a given FOODE, then the corresponding integrating factor can be written as: $R = \prod_i f_i^{n_i}$ where f_i are irreducible polynomials and n_i are nonzero rational numbers. One might then ask whether there are cases where the solution is non-elementary and yet one can find an integrating factor of the form just described. As we shall see shortly, through the simple illustrative example below, it is actually the case.

Consider:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^2x^2 - y^2 + 4}{4xy(x+1)(x-1)}. \tag{7}$$

For this FOODE, we have

$$M = 3x^2y^2 - y^2 + 4 \qquad N = 4xy(x+1)(x-1) \tag{8}$$

and so

$$D = (3x^2y^2 - y^2 + 4)\partial_x + 4xy(x+1)(x-1)\partial_y. \tag{9}$$

6 For FOODEs such as (1), where M and N are infinitely differentiable functions, there always exist such functions.
7 In other words, f_i is a factor of $D[f_i]$.

In order to follow the PS approach we look for all possible irreducible polynomials f_i such that $D[f_i]/f_i$ is a polynomial. Up to polynomials of degree 1, we find for this case: $f_1 = x$, $f_2 = x + 1$ and $f_3 = 1 - x$. Solving (6), we get: $n_1 = n_2 = n_3 = -3/2$. So, we have an integrating factor for (7) given by

$$R = \prod_i f_i^{n_i} = x^{-3/2}(x+1)^{-3/2}(1-x)^{-3/2} \tag{10}$$

and the corresponding solution is

$$\int \frac{3y^2x^2 - y^2 + 4}{x^{3/2}(x+1)^{3/2}(1-x)^{3/2}}\,\mathrm{d}x - \_C1 = 0. \tag{11}$$

The above integral can be expressed in terms of elliptic functions (which are non-elementary).

As previously stated, the PS approach assures that, if the solution is expressible in terms of elementary functions, there will be an integrating factor of the form $\prod_i f_i^{n_i}$. However, it is important to emphasize, the PS result *does not say* that, if the integrating factor is of the form $\prod_i f_i^{n_i}$, the solution will be elementary! The above FOODE is an example of that.

Let us now analyse another FOODE presenting a Liouvillian solution that is non-elementary. In this case, the application of the PS approach is not successful and we will show below why that had to be so. The FOODE is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + yx + x - 1 \tag{12}$$

equation I.18 of the standard testing ground for ODE solvers by Kamke [12].

The solution for this FOODE is

$$\int \frac{\mathrm{e}^{x^2/2-2x}(y^2 + xy + x - 1)}{(y+1)^2}\,\mathrm{d}x. \tag{13}$$

The above integral can be expressed in terms of the error function (which is non-elementary). Let us have a look at the integrating factor for this FOODE to understand why the PS approach fails for this case

$$R = \frac{\mathrm{e}^{x^2/2-2x}}{(y+1)^2}. \tag{14}$$

Since the standard PS procedure constructs integrating factor candidates from polynomials in the variables (x, y), one can see that, since the integrating factor on (14) presents the exponential $\mathrm{e}^{x^2/2-2x}$, it will never be found by the PS method.

In this section, we have pointed out that the FOODEs that present non-elementary Liouvillian solutions can be broadly divided into two groups; those presenting integrating factors of the form $\prod_i f_i^{n_i}$ (in which cases the PS method will be successful) and those with integrating factors that cannot be put on the general form just presented. In the next section, we will start from a result due to Singer [11] and demonstrate a theorem concerning the general structure of the integrating factor for LFOODEs.

4. Concerning the general structure of the integrating factor for LFOODEs

In this section, we are going to present a result concerning the general form for the integrating factor for FOODEs with Liouvillian solutions (LFOODEs). We are going to start our analysis from the following result (due to Singer [11]):

Singer's result. *If we have a LFOODE of the form* $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$*, where M and N are polynomials in* (x, y)*, then it presents an integrating factor of the form* $\mathrm{e}^{(\int(U\,\mathrm{d}x + V\,\mathrm{d}y))}$, *where U and V are rational functions of (x, y) with $U_y = V_x$ so that this latter line integral is well defined*[8].

From this, we extracted the following theorem:

Theorem. *If a LFOODE is of the form $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$, where M and N are polynomials in (x, y), then it presents an integrating factor that can be put in the form:*

$$R = \mathrm{e}^{r_0(x,y)} \prod_{i=1}^{n} p_i(x, y)^{c_i} \tag{15}$$

where r_0 is a rational function of (x, y), the p_i are irreducible polynomials in (x, y) and the c_i are constants.

To prove this theorem we need the following lemma.

Lemma. If ω is a function of (x, y) such that both partial derivatives (ω_x, ω_y) are rational functions of (x, y), then ω can be written as

$$\omega = r_0 + \sum_i \alpha_i \ln(r_i) \tag{16}$$

where the r_i are rational functions of (x, y) (including $i = 0$), and α_i are constants.

Proof of the lemma. Let us write ω in the following form[9]:

$$\omega = f(x) + g(y) + h(x, y) \tag{17}$$

where f is a function of x only, g is a function of y only and h is such that $\int h_x \,\mathrm{d}x = \int h_y \,\mathrm{d}y = h$. So, we have that

$$\omega_x = h_x + \frac{\mathrm{d}f}{\mathrm{d}x} \tag{18}$$

$$\omega_y = h_y + \frac{\mathrm{d}g}{\mathrm{d}y} \tag{19}$$

$$\int \omega_x \,\mathrm{d}x = \int h_x \,\mathrm{d}x + \int \frac{\mathrm{d}f}{\mathrm{d}x} \,\mathrm{d}x = h + f \tag{20}$$

$$\int \omega_y \,\mathrm{d}y = \int h_y \,\mathrm{d}y + \int \frac{\mathrm{d}g}{\mathrm{d}y} \,\mathrm{d}y = h + g \tag{21}$$

From well-known results concerning formal integration [13], we have that if $\rho(u)$ is a rational function of u, then

$$\int \rho(u) \,\mathrm{d}u = \rho_0(u) + \sum_i \kappa_i \ln(\rho_i(u)) \tag{22}$$

where ρ_i $(i = 0, \ldots)$ are rational functions of u, and $\kappa_i (i = 1, \ldots)$ are constants.

Therefore, if we have a function $F(u)$ such that its derivative $(\mathrm{d}F/\mathrm{d}u)$ is a rational function of u, then $F(u)$ can be written as

$$\int \frac{\mathrm{d}F}{\mathrm{d}u} \,\mathrm{d}u = F = F_0 + \sum_i C_i \ln(F_i) \tag{23}$$

where the F_i are rational functions of u (including $i = 0$), and C_i are constants.

[8] From now on, F_u will mean $\partial_u F$.

[9] Any analytical function can be written in this form. As will become clear soon, this is convenient for our demonstration.

Now suppose that the hypotheses of the lemma are satisfied. Then, since ω_x is a rational function of (x, y), then (see (18)) h_x is a rational function of (x, y) and $\mathrm{d}f/\mathrm{d}x$ is a rational function of x. Then, from (23), we have

$$\int h_x\,\mathrm{d}x = h = h_0(x, y) + \sum_i c_i(y)\ln(h_i(x, y)) \tag{24}$$

where h_i are functions of (x, y), rational in x, and c_i do not depend on x, and

$$\int \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f = f_0(x) + \sum_j a_j \ln(f_j(x)) \tag{25}$$

where f_j are rational functions of x only, and a_j are constants.

From the results in [13], in principle, h_i and c_i could be algebraic functions[10] of y. However, ω_y is a rational function of (x, y) and implies (see (19)) that h_y is a rational function of (x, y) and $\mathrm{d}g/\mathrm{d}y$ is a rational function of y only. So, differentiating (24) with respect to y, we get

$$h_y = h_{0y} + \sum_i c_i \frac{h_{iy}}{h_i} + \sum_i \frac{\mathrm{d}c_i}{\mathrm{d}y}\ln(h_i). \tag{26}$$

Since h_y is a rational function of (x, y), the logarithmic terms must vanish, leading to (since they cannot cancel out) $\mathrm{d}c_i/\mathrm{d}y = 0 \to c_i$ are constants. With that in mind, integrating (26) with respect to y we have

$$\int h_y\,\mathrm{d}y = h = h_0(x, y) + \sum_i c_i \ln(h_i(x, y)) \tag{27}$$

where c_i are constants. From (23), we conclude that h_i must be rational functions of y rather then algebraic. So, the h_i $(i = 0, 1, \ldots)$ are rational functions of (x, y).

Since $\mathrm{d}g/\mathrm{d}y$ is a rational function of y, we have

$$\int \frac{\mathrm{d}g}{\mathrm{d}y}\,\mathrm{d}y = g = g_0(y) + \sum_k b_k \ln(g_k(y)) \tag{28}$$

where b_k are constants and g_k are rational functions of y.

Finally, since $\omega = f(x) + g(y) + h(x, y)$ we may conclude that ω can be written as in (16) as we wanted to demonstrate. □

Using Singer's results mentioned above and the lemma we have just demonstrated, we can prove the theorem.

Proof of the theorem. Consider that the hypotheses in Singer's result are satisfied. Since $U_y = V_x$, we can choose a function $\omega(x, y)$ such that $\mathrm{d}\omega = U\,\mathrm{d}x + V\,\mathrm{d}y$, i.e., $\omega_x = U$ and $\omega_y = V$. So, using Singer's result, it is straightforward to see that

$$R = \mathrm{e}^{\int (U\,\mathrm{d}x + V\,\mathrm{d}y)} = \mathrm{e}^{\int \mathrm{d}\omega} = \mathrm{e}^{\omega}. \tag{29}$$

Note that, from the hypotheses in Singer's result, $\omega_x = U$ and $\omega_y = V$ are rational functions of (x, y). So, using the lemma above, we can write

$$R = \mathrm{e}^{\omega} = \mathrm{e}^{r_0 + \sum_i \alpha_i \ln(r_i)} = \mathrm{e}^{r_0}\prod_i (r_i)^{\alpha_i} \tag{30}$$

[10] For a formal definition of algebraic functions, see [13].

where the r_i are rational functions of (x, y) (including $i = 0$), and α_i are constants. Since the r_i are rational, we can write $\prod_i (r_i)^{\alpha_i}$ as a product of powers of irreducible polynomials. So, we can finally write

$$R = \mathrm{e}^{r_0} \prod_{i=1}^{n} p_i^{\beta_i} \tag{31}$$

where r_0 is a rational function of (x, y), the p_i are irreducible polynomials in (x, y) and the β_i are constants, thus completing the proof. □

Note that, if r_0 is constant ($r_0 = k$), we have from (31) that $R = \mathrm{e}^k \prod_{i=1}^n p_i^{\beta_i} = K \prod_{i=1}^n p_i^{\beta_i}$ implying that $\prod_{i=1}^n p_i^{\beta_i}$ is also an integrating factor, which is equivalent to the general expression used in the PS method (see section (2)). Therefore, our result for the general structure for the integrating factor extends our knowledge of the general structure for the integrating factor for FOODEs with Liouvillian solutions. In what follows, based on this result, we are going to present a method to find R for a class of FOODEs where r_0 is not a constant and so the PS method is not applicable.

5. Finding the integrating factor

In this section, using the result just presented above (31) and making a conjecture, we are going to present a method of finding the integrating factor for a class of LFOODEs using a procedure that extends the PS method. We then present some examples of its applicability.

5.1. Introduction

By using R as given on (31) and (4), one finds

$$D[r_0(x, y)] + \sum_i \frac{c_i D[p_i]}{p_i} = -(\partial_x N + \partial_y M). \tag{32}$$

In the PS method, the analogous to the above equation is (6):

$$\sum_i \frac{n_i D[f_i]}{f_i} = -(\partial_x N + \partial_y M).$$

Note that the main difference among these equations is the presence of the 'extra' term $D[r_0]$ on (32). For (6), using the fact that M and N are polynomials in (x, y), one can prove that f_i are eigenpolynomials of the operator D, i.e., $D[f_i]/f_i$ is a polynomial [6]. Regarding (32), if $D[r_0]$ is a polynomial, by the same line of reasoning, one can conclude that p_i are eigenpolynomials of the operator D, i.e., $D[p_i]/p_i$ is a polynomial. In the following, we will show that, if $D[r_0]$ is a polynomial, we are able to build a method to tackle a class of FOODEs where the PS method is not applicable. So, we are going to make the following conjecture.

Conjecture. *If* $R = \mathrm{e}^{r_0} \prod_{i=1}^n p_i^{c_i}$, *where* r_0 *is a rational function of* (x, y), *the* p_i *are irreducible polynomials in* (x, y) *and the* c_i *are constants, is an integrating factor for a LFOODE of the form* $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$, *where* M *and* N *are polynomials in* (x, y), *then* $D \equiv N\partial_x r_0 + M\partial_y r_0$ *is a polynomial.*

Let us now introduce the method.

5.2. The method

The method we are now going to introduce enables us to successfully deal with the cases where $r_0(x, y) = r(x) + s(y)$, where both r and s are rational functions. As we have previously mentioned, for this case the PS method has no hope of finding the integrating factor (and consequently integrate the FOODE).

The LFOODEs that correspond to the above condition define the class of equations solved by our method.

The first step of our method is that, since we made a hypothesis implying that the p_i are eigenpolynomials of the D operator, we have: $D[p_i] = g_i p_i$, where g_i are polynomials called eigenvalues of D. We can then calculate all the p_i and associated g_i up to a given degree, in the same fashion as the PS method, for the LFOODE that we want to solve. We then use the knowledge of this set of p_i and g_i in the next steps of the method.

Since in this section we are interested in presenting the most effective method possible for solving FOODEs, we will actually divide the next steps of our method into three subcases:

1. Looking for r_0 of the type $r_0 = r_0(x)$
2. Looking for r_0 of the type $r_0 = r_0(y)$
3. Looking for r_0 of the type $r_0 = r(x) + s(y)$[11]

where all of the above functions are rational. Why would we do so? The point is that, as we shall make clear below, the tackling of the 'full' case $r_0(x, y) = r(x) + s(y)$ is much more involved than the tackling of the 'partial' cases ($r_0(x, y) = r(x)$), $r_0(x, y) = s(y)$). So, since when tackling a FOODE we do not know *a priori* what is the integrating factor (otherwise we would have solved the problem already), we could be facing a FOODE for which the corresponding integrating factor has $r(x) \neq 0$ and $s(y) = 0$ or $r(x) = 0$ and $s(y) \neq 0$ and so it is worth (for reasons of time consumption) pursuing first the partial cases just described (items 1 and 2), before trying the 'full' case (item 3).

Bearing that in mind, we are going now to present the next steps of the method. These steps depend on the subcases (items 1, 2 or 3) which we are pursuing. As in the case of the PS method, our method is of a semi-decision nature (see section 1), i.e. there is an uncertainty concerning the degree of the polynomials involved. So, in principle, one could be doing calculations *ad infinitum* in each of the three subcases (see below). The decision about whether to keep on doing calculations within the same subcase or trying the next one is an open matter. Let us now introduce the next steps of our method for each of the three subcases:

5.2.1. Subcase $r_0 = r_0(x)$. Reminding the reader that the D operator is defined by $D \equiv N\partial_x + M\partial_y$, (32) will then become

$$N\frac{\mathrm{d}r_0(x)}{\mathrm{d}x} + \sum_i \frac{c_i D[p_i]}{p_i} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right). \tag{33}$$

We can then write (33) as

$$N\frac{\mathrm{d}r_0(x)}{\mathrm{d}x} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) - \sum_i c_i g_i \tag{34}$$

leading to

$$\frac{\mathrm{d}r_0(x)}{\mathrm{d}x} = -\frac{N_x + M_y + \sum_i c_i g_i}{N}. \tag{35}$$

[11] Full case, much more involved than the first two partial ones.

In order to find $r_0(x)$ we need first to integrate (35):

$$r_0(x) = -\int \frac{N_x + M_y + \sum_i c_i g_i}{N}\,\mathrm{d}x. \qquad (36)$$

Since r_0 is a *rational* function of x only, we can determine the c_i by imposing that the derivative of the right-hand side of (36) with respect to y be equal to zero and that the logarithmic terms that might arise from the integration vanish. If we manage to find a set of c_i satisfying these conditions, up to the degree we are considering, we will have found the integrating factor for the LFOODE. Otherwise, we then increase the degree of the p_i and try again, until we succeed, in the same manner as in the PS method. See an example of this subcase in section 5.3.1.

5.2.2. Subcase $r_0 = r_0(y)$. Analogously, for this case, (32) will become

$$M\frac{\mathrm{d}r_0(y)}{\mathrm{d}y} + \sum_i \frac{c_i D[p_i]}{p_i} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right). \qquad (37)$$

Following a procedure similar to the one used in section 5.3.1, we can write $r_0(y)$ as

$$r_0(y) = -\int \frac{N_x + M_y + \sum_i c_i g_i}{M}\,\mathrm{d}y. \qquad (38)$$

Since r_0 is a *rational* function of y only, we can determine the c_i by imposing that the derivative of the right-hand side of (38) with respect to x be equal to zero and that the logarithmic terms that might arise from the integration vanish. In the same way, if we manage to find a set of c_i satisfying these conditions, up to the degree we are considering, we will have found the integrating factor for the LFOODE. Otherwise, we then increase the degree of the p_i and try again, until we succeed, in the same manner as in the PS method. See an example of this subcase in section 5.3.2.

5.2.3. Subcase $r_0 = r(x) + s(y)$. For this case, (32) becomes

$$N\frac{\mathrm{d}r(x)}{\mathrm{d}x} + M\frac{\mathrm{d}s(y)}{\mathrm{d}y} + \sum_i \frac{c_i D[p_i]}{p_i} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right). \qquad (39)$$

Dividing the equation above by N and isolating $\frac{\mathrm{d}r(x)}{\mathrm{d}x}$, one obtains

$$\frac{\mathrm{d}r(x)}{\mathrm{d}x} = -\frac{M}{N}\frac{\mathrm{d}s(y)}{\mathrm{d}y} - \frac{N_x + M_y + \sum_i c_i g_i}{N}. \qquad (40)$$

Integrating both sides of the equation, with respect to x:

$$r(x) = -\frac{\mathrm{d}s(y)}{\mathrm{d}y}\int \frac{M}{N}\,\mathrm{d}x - \int \frac{N_x + M_y + \sum_i c_i g_i}{N}\,\mathrm{d}x. \qquad (41)$$

We can then isolate $\frac{\mathrm{d}s(y)}{\mathrm{d}y}$ in (41) to find

$$\frac{\mathrm{d}s(y)}{\mathrm{d}y} = \frac{\int \frac{N_x + M_y + \sum_i c_i g_i}{N}\,\mathrm{d}x - r(x)}{\int \frac{M}{N}\,\mathrm{d}x}. \qquad (42)$$

An analogous procedure leads to

$$\frac{\mathrm{d}r(x)}{\mathrm{d}x} = \frac{\int \frac{N_x + M_y + \sum_i c_i g_i}{M}\,\mathrm{d}y - s(y)}{\int \frac{N}{M}\,\mathrm{d}y}. \qquad (43)$$

The c_i remain to be determined. Once again, we still have to impose that, by construction, $r(x)$ and $\mathrm{d}r(x)/\mathrm{d}x$ do not depend on y and, analogously, $s(y)$ and $\mathrm{d}s(y)/\mathrm{d}y$ do not depend on x. By imposing this, and that both $r(x)$ and $s(y)$ (and consequently $\mathrm{d}r(x)/\mathrm{d}x$ and $\mathrm{d}s(y)/\mathrm{d}y$) are *rational* functions, we may hope to find a set of suitable values for c_i. If that is the case, we would have found $r(x)$ and $s(y)$ and, consequently, the integrating factor. If this fails, we have to increase the degree of the p_i and try all over again. See an example of this subcase in section 5.3.3.

5.3. Examples

In this section, we are going to present one example of application of our method for each of the subcases mentioned above.

5.3.1. $r_0 = r_0(x)$. Consider the LFOODE (I.129 in [12]):

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + y(y-x) = 0. \tag{44}$$

For this equation, up to degree 1, we have that the eigenpolynomials (with the associated eigenvalues) are

- $p_1 = y, g_1 = (x - y)$
- $p_2 = (x + 1), g_2 = 1.$

So, (36) becomes

$$r_0(x) = ((2 + c_1)\, y - c_2 + c_1) \ln(x + 1) - x - c_1 x. \tag{45}$$

Imposing that r_0 is a rational function of x will lead to

$$c_1 = -2 \qquad c_2 = -2 \tag{46}$$

and, consequently, to (using (31)):

$$r_0(x) = x \quad \rightarrow \quad R = \frac{\mathrm{e}^x}{y^2(x+1)^2} \tag{47}$$

5.3.2. $r_0 = r_0(y)$. Consider the LFOODE (I.235 in [12]):

$$(xy + a)\frac{\mathrm{d}y}{\mathrm{d}x} + by = 0. \tag{48}$$

For this equation, up to degree 1, we have that the eigenpolynomials (with the associated eigenvalues) are

- $p_1 = y, g_1 = -b.$

So, (36) becomes

$$r_0(y) = (-1 - c_1) \ln(y) + \frac{y}{b} \tag{49}$$

Imposing that r_0 is a rational function of y will lead to

$$c_1 = -1 \tag{50}$$

and, consequently, to (using (31)):

$$r_0(y) = \frac{y}{b} \quad \rightarrow \quad R = \frac{\mathrm{e}^{\frac{y}{b}}}{y} \tag{51}$$

5.3.3. $r_0 = r(x) + s(y)$. Consider the Abel LFOODE of the first kind:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2(y+x-1)}{x^2}. \tag{52}$$

For this equation, up to degree 1, we have that the eigenpolynomials (with the associated eigenvalues) are

- $p_1 = x$, $g_1 = x$,
- $p_2 = y$, $g_2 = y^2 + yx - y$,
- $p_3 = x + y$, $g_3 = x - y + y^2$.

So, from (43) we get

$$\frac{\mathrm{d}r(x)}{\mathrm{d}x} = -\frac{\left((-c_2-2)\,x^2 + (2c_3+2c_2+6+c_1)\,x - c_3 - c_2 - 2\right) y\ln(y)}{x^2\,(x-1+\ln(y)y - \ln(y+x-1)y)} - \frac{\left((-c_3-1)\,x^2 - c_1x - 1\right) y\ln(y+x-1)}{x^2(x-1+\ln(y)y-\ln(y+x-1)y)} - \frac{(-x^2+2x-1)ys(y) + (c_3+2+c_1)\,x^2 + (-c_1-c_3-2)\,x}{x^2\,(x-1+\ln(y)y-\ln(y+x-1)y)} = 0 \tag{53}$$

leading to

$$\left(-\frac{\mathrm{d}r(x)}{\mathrm{d}x}x^2y + \left((-c_3-1)\,x^2 - c_1x - 1\right)y\right)\ln(y+x-1) + \left(\frac{\mathrm{d}r(x)}{\mathrm{d}x}x^2y + \left((-c_2-2)\,x^2 + (2c_3+2c_2+6+c_1)x - c_3 - c_2 - 2\right)y\right)\ln(y) + (x^3-x^2)\frac{\mathrm{d}r(x)}{\mathrm{d}x} + (-x^2+2x-1)ys(y) + (c_3+2+c_1)\,x^2 + (-c_1-c_3-2)\,x = 0. \tag{54}$$

Since $\mathrm{d}r(x)/\mathrm{d}x$ and $s(y)$ are rational functions, in the above equation the coefficients of the logarithmic terms must vanish. Therefore:

$$-\frac{\mathrm{d}r(x)}{\mathrm{d}x}x^2y + \left((-c_3-1)\,x^2 - c_1x - 1\right)y = 0 \tag{55}$$

$$\frac{\mathrm{d}r(x)}{\mathrm{d}x}x^2y + \left((-c_2-2)\,x^2 + (2c_3+2c_2+6+c_1)\,x - c_3 - c_2 - 2\right)y = 0. \tag{56}$$

Solving (55) and (56) for $\mathrm{d}r(x)/\mathrm{d}x$ and integrating in x, one gets

$$r(x) = -x - c_3x + x^{-1} - c_1\ln(x)) \tag{57}$$

$$r(x) = (-2c_3 - 2c_2 - 6 - c_1)\ln(x) + c_2x + 2x - 2x^{-1} - \frac{c_2}{x} - \frac{c_3}{x}. \tag{58}$$

Imposing that the logarithmic terms vanish and that the right-hand sides of (57) and (58) are equal, we get

$$\{c_1 = 0, c_2 = -c_3 - 3, c_3 = c_3\}. \tag{59}$$

Using these in (39), solving the resulting equation for $\mathrm{d}s(y)/\mathrm{d}y$ and imposing that it cannot depend on x, we have

$$(1+c_3)x^2 + ((2+2c_3)\,y - 2 - 2c_3)\,x + (1+c_3)\,y^2 + (-2-2c_3)\,y + 1 + c_3 = 0 \tag{60}$$

implying that $c_3 = -1$. So, using (59):

$$\{c_1 = 0, c_2 = -2, c_3 = -1\}. \tag{61}$$

Substituting (61) into (57), we get $r(x) = 1/x$. Using this in (54) and solving $s(y)$, we have finally (using (31)):

$$r(x) = \frac{1}{x} \qquad s(y) = \frac{1}{y} \quad \rightarrow \quad R = \frac{\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}}{y^2(x+y)}. \tag{62}$$

6. Conclusion

In this paper, we have analysed why the PS method, which is designed to deal with the first-order ordinary differential equations with elementary solutions, can solve some first-order ordinary differential equations with non-elementary (Liouvillian) solutions. From that analysis, we have shown that, for a LFOODE of the type given by (1), the integrating factor can be written as

$$R = \mathrm{e}^{r_0} \prod_{i=1}^{n} p_i^{\beta_i} \tag{63}$$

where r_0 is a rational function of (x, y), the p_i are irreducible polynomials in (x, y) and the β_i are constants.

Based on that, we have presented a method which solves a class of LFOODEs out of the scope of the PS method. Although our method deals with a restricted class of LFOODEs (of type (1) and among those the ones defined by the three subcases considered in section 5.2), we believe the method to be a valid contribution since, for example, it solves LFOODEs that 'escape' from powerful solvers[12], using any method of solution (the example in section 5.3.3 is such a case). At the time of writing this paper, the conjecture we have herein used was not proved. That now has been completed [14].

We hope to present results concerning the extension of our method to deal with a larger class of LFOODEs in the near future. For example, considering $r_0(x, y)$ to be a general rational function or LFOODEs with $M(x, y)$ and/or $N(x, y)$ not restricted to be polynomials, etc.

References

[1] Stephani H 1989 *Differential Equations: Their Solution Using Symmetries* ed M A H MacCallum (New York: Cambridge University Press)

[2] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (*Applied Mathematical Science* vol 81) (Berlin: Springer)

[3] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[4] Cheb-Terrab E S, Duarte L G S and da Mota L A C P 1997 Computer algebra solving of first order ODEs using symmetry methods *Comput. Phys. Commun.* **101** 254

[5] Cheb-Terrab E S, Duarte L G S and Mota da L A C P 1998 Computer algebra solving of second order ODEs using symmetry methods *Comput. Phys. Commun.* **108** 90

[6] Prelle M and Singer M 1983 Elementary first integral of differential equations. *Trans. Am. Math. Soc.* **279** 215

[7] Shtokhamer R 1988 Solving first order differential equations using the Prelle–Singer algorithm *Technical Report* 88-09 Center for Mathematical Computation, University of Delaware

[8] Collins C B 1993 Algebraic invariants curves of polynomial vector fields in the plane *Preprint* University of Waterloo, Canada

Collins C B 1993 Quadratic vector fields possessing a centre *Preprint* University of Waterloo, Canada

[9] Man Y K and MacCallum M A H 1996 A rational approach to the Prelle–Singer algorithm *J. Symb. Comput.* **11** 1–11 and references therein

[10] Duarte L G S, Duarte S E S, da Mota L A C P and Skea J E F 2001 Solving second order ordinary differential equations by extending the Prelle–Singer method *J. Phys. A: Math. Gen.* **34** 3015–24

[11] Singer M 1992 Liouvillian first integrals *Trans. Am. Math. Soc.* **333** 673–88

[12] Kamke E 1959 *Differentialgleichungen: Lösungsmethoden und Lösungen* (New York: Chelsea)

[13] Davenport J H, Siret Y and Tournier E 1993 *Computer Algebra: Systems and Algorithms for Algebraic Computation* (London: Academic)

[14] Duarte L G S, Duarte S E S and da Mota L A C P 2002 Analysing the structure of the integrating factor for first order differential equations with Liouvillian functions in the solutions *J. Phys. A: Math. Gen.* **35** 1001–6

[12] For example, the **dsolve** command in MAPLE, release 5.1.